Multi-scale Finite Element Method for incompressible flow in Perforated Domain

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Multi-scale problems arise in numerous engineering fields such as reservoir engineering, flows through fractured porous media, flows in nuclear reactor cores, etc. In these media with many obstacles of various sizes, the macroscopic flow is directly influenced by local phenomena occurring at the finest scales. Thus, computing numerically such flows requires a very fine mesh to resolve all the details. Despite the continuous increase in computer resources, these are insufficient to perform classical finite element simulations with an accuracy allowing correct resolution of the finest scales of the flow. To overcome this limitation, various multi-scale methods have been developed to attempt to resolve scales below the coarse mesh scale by incorporating local computations into a global problem which is defined only on a coarse mesh. Among the many multi-scale approaches that have been proposed in the literature, we can mention the Heterogeneous Multi-scale Method (HMM) [1], the Local Orthogonal Decomposition (LOD) [2] or the Multi-scale Finite Element Method (MsFEM). In this contribution, we focus on the Multi-scale Finite Element Method.

The Multi-scale Finite Element Method uses a coarse mesh on which one defines basis functions which are no longer the classical polynomial basis functions of finite elements, but which solve fluid mechanics equations on the elements of the coarse mesh. These functions are themselves numerically approximated on a fine mesh considering all the geometric details, which gives the multi-scale aspect of this method.

Based on the work of [3, 4], we propose to develop an enriched non-conforming Multi-scale Finite Element Method (MsFEM) to solve viscous incompressible flow in heterogeneous media. Our MsFEM is in the vein of the classical non-conforming Crouzeix-Raviart finite element method with high-order weighting functions. At the theoretical level, in order to complete the work of [3], we perform a rigorous study of this MsFEM in two and three dimensions to solve the Stokes or the Oseen equations. We show the well-posedness of the discretized local problems firstly for a family of non-conforming finite elements of arbitrary order on triangles presented in [5], and secondly for a new family of finite elements that we have developed in three dimensions. In addition, we quantify the error between the MsFEM and the exact solution for the global Stokes problem in perforated domain using homogenisation theory.

At the numerical level, we implement the Multi-Scale Finite Element Methods developed, in two and three dimensions, up to the order two, in a parallel framework using PETSc in FreeFEM [6] : the basis functions being independent of each other, their approximations as well as the assembly of the macroscopic problem can be carried out in parallel. We compare the MsFEM approximations with reference solutions obtained by performing the simulations on a fine grid with a classical Finite Element Method. Besides, we compare the Galerkin and the Petrov-Galerkin approaches for solving the Oseen equations.

The perspective of this work is now to develop a methodology to solve the Navier-Stokes equations with multi-scale basis functions. In parallel, to complete the study of our MsFEM for the Stokes equations, we are investigating on an a posteriori error estimate.

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