A Shape Characterization of Quadrics

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Let $D \subseteq \mathbb{R}^2$ be a domain and let $S : D \to \mathbb{R}^3$ be a surface of class C^3 with a parametrization $\{S(u,v),(u,v) \in D\}$ and a unit normal vector field $\{N(u,v),(u,v) \in D\}$.There are two important types of surfaces which are associated to S. The surfaces of the first type form a one-parameter set of parallel surfaces $\{S_d, d \in \mathbf{R}\}$ with parameterizations $S_d(u,v) = S(u,v) + dN(u,v), (u,v) \in D$. The surfaces of the second type are two focal surfaces S_{f1} and S_{f2} of S parameterized by

$$S_{fi}(u,v) = S(u,v) + \frac{1}{f_i(u,v)}N(u,v), \quad (u,v) \in D,\, i = 1,2,$$

where $f_1(u,v)$ and $f_2(u,v)$ are the principal curvatures at the point S(u,v) of S. Any Euclidean motion of $\mathbb{R}^3$ preserves the pairs $\big(S,S_d\big)$, $\big(S,S_{f1}\big)$ and $\big(S,S_{f2}\big)$. Therefore, there is a large number of papers dealing with construction and investigation of S_d, S_{f1} and S_{f2} for different classes of surfaces in $\mathbb{R}^3$.

Hans Hagen and Stefanie Hahmann introduced and studied particular cases of another associated surface S_g to S which is called a generalized focal surface of S. The surface S_g is defined by the vector parametric equation

$$S_g(u,v) = S(u,v) + g(u,v)N(u,v), \quad (u,v) \in D,$$

where $g(u,v)$ is a function of $f_1(u,v)$ and $f_2(u,v)$.

The smallest extension of the group of Euclidean motions of $\mathbb{R}^3$ is the group of the direct similarities of $\mathbb{R}^3$ denoted by $Sim^+(\mathbb{R}^3)$. Any element of the last group is an affine transformation that preserves the orientation and the angles. The direct similarities preserve the shape of geometric objects.

In previous author's paper it was proved that the functions $g_1(u,v)$ and $g_2(u,v)$ on a surface S with a non-vanishing Gaussian curvature K and a non-vanishing mean curvature H given by $g_1(u,v) = H/K = (f_1(u,v)+f_2(u,v))/(2f_1(u,v)f_2(u,v))$ and $g_2(u,v) = 1/H = 2/(f_1(u,v)+f_2(u,v))$ are similarity invariant functions of S. Then, for a surface S with a non-vanishing Gaussian curvature K and a non-vanishing mean curvature H, there are determined two generalized focal surfaces

$$S_{g1}(u,v) = S(u,v) + g_1(u,v)N(u,v), \quad (u,v) \in D,$$

and

$$S_{g2}(u,v) = S(u,v) + g_2(u,v)N(u,v), \quad (u,v) \in D.$$

Any element σ of $Sim^+(\mathbb{R}^3)$ preserves the pairs $\big(S,S_{g1}\big)$, and $\big(S,S_{g2}\big)$. This means that $\sigma(S_{g1})$ is the generalized focal surface of $\sigma(S)$ of the same kind as $\big(S,S_{g1}\big)$.

In this poster presentation it is computed similarity invariant functions $g_1(u,v)$ and $g_2(u,v)$ for an ellipsoid, a hyperboloid of one sheet,a hyperboloid of two sheets, an elliptic paraboloid and a hyperbolic paraboloid. For the same quadrics, the parametric equations of the generalized focal surfaces S_{g1} and S_{g2} are calculated. Both similarity invariant functions and the considered generalized focal surfaces are related to the shape of the mentioned quadrics.