Harder-Narasimhan and persistence module: multiple central charges

Emile Jacquard
University of Oxford, UK
emile.jacquard@maths.ox.ac.uk

The work presented here is best illustrated by Sections 3, 5.1 and 6 of the preprint `https://arxiv.org/abs/2303.16075`. This was written by Ulrike Tillmann, Vidit Nanda, Marc Fersztand and Emile Jacquard, all affiliated with the University of Oxford.

The Harder-Narasimhan type of a quiver representation is a discrete invariant parameterised by a real-valued function (called a central charge) defined on the vertices of the quiver. We introduce the skyscraper invariant, which amalgamates the HN types along central charges supported at single vertices, and generalise the rank invariant from multiparameter persistence modules to arbitrary quiver representations. Our main results are as follows: (1) we show that the skyscraper invariant is strictly finer than the rank invariant in full generality, (2) we show that although no single central charge is complete for nestfree ladder persistence modules, a finite set of central charges is complete.

A central charge on a quiver Q is a function $\alpha\colon Q_0 \mapsto \mathbb{R}$. The α-slope of a representation V of Q is the ratio

$$\mu_\alpha(V) = \frac{\sum_{x\in Q_0} \alpha(x)\cdot \dim V_x}{\sum_{x\in Q_0} \dim V_x}.$$

V is α-semistable if all its subrepresentations have smaller α-slope. Every nonzero representation V admits a unique *Harder-Narasimhan* filtration of finite length

$$0 = HN_\alpha^0(V) \subsetneq HN_\alpha^1(V) \subsetneq \cdots \subsetneq HN_\alpha^{n-1}(V) \subsetneq HN_\alpha^n(V) = V$$

whose successive quotients $S^i := HN_\alpha^i(V)/HN_\alpha^{i-1}(V)$ are α-semistable and satisfy $\mu_\alpha(S^i) > \mu_\alpha(S^{i-1})$ for all i. The main discrete invariant of interest is the **HN type** of V along α

$$\mathbf{T}[\alpha][V] = \left(\underline{\dim}\, S^1, \underline{\dim}\, S^2 \ldots, \underline{\dim}\, S^n\right).$$

Given a collection of central charges, the collection of associated HN types is also a discrete invariant. Consider, for each vertex x, the central charge $\delta_x : Q_0 \to \mathbb{R}$ which maps x to 1 and all other vertices to 0. The skyscraper invariant δ_V is the collection of HN types $\mathbf{T}[\delta_x][V]$ indexed by the vertices of Q. Our main result is

Theorem 1: The skyscraper invariant is finer than the rank invariant on *Rep*(Q) for any finite quiver Q.

We prove this using the spanning subrepresentation of V at a vertex x — this is defined up to isomorphism as the smallest subrepresentation $\langle V_x\rangle \subset V$ containing V_x. The function $\rho_V : Q_0 \times Q_0 \to \mathbb{N}$ that sends each (x, y) to the dimension of $\langle V_x\rangle_y$ vastly generalises the rank invariant of Carlsson and Zomorodian.

Next, we consider nest-free ladders (commuting representations of the ladder quiver with no nested bars in the top and bottom rows), and show that no single central charge yields a complete invariant, but that we may find a collection of central charges whose associated HN-types yield a complete invariant.

Theorem 2: There is no complete central charge on the category of nestfree ladder persistence modules of length $\ell \geq 4$; however, for all ℓ there exists a finite set $A = A(\ell)$ of central charges which is complete on this category.

Joint work with Ulrike Tillmann, Vidit Nanda and Marc Fersztand.