Transverse Symplectic Foliation Structure of Souriau Dissipative Statistical Mechanics with Entropy as Casimir Function

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Foliation theory is a natural generalized qualitative theory of differential equations, initiated by H. Poincaré, and developed by C.Ehresmann and G. Reeb, with contribution by A. Haefliger, P. Molino, B.L. Reinhart. Riemannian foliations generated by metric functions were developed by Ph. Tondeur. Notion of foliation in thermodynamics appears as soon as 1900 C. Carathéodory paper where horizontal curves roughly correspond to adiabatic processes, performed in the language of Carnot cycles. The properties of the couple of Poisson manifolds was also previously explored by C. Carathéodory in 1935, under the name of "function groups, polar to each other", where he observed that the two families of differentiable functions formed by the first integrals of F (a completely integrable vector subbundle of TM) and its orthogonal orthF, respectively, called "function groups", are "polar" of the other. This seminal work of C. Caratheodory leads to the concept of a Poisson structure which was first defined and treated in depth by A. Lichnerowicz and independently by A. Kirillov. We introduce a symplectic bifoliation model of Statistical Mechanics, Information Geometry and Heat Theory based on Jean-Marie Souriau's Lie Groups Thermodynamics to describe transverse Poisson structure of metriplectic flow for dissipative phenomena. This model gives a cohomological characterization of Entropy, as an invariant Casimir function in coadjoint representation. The dual space of the Lie algebra foliates into coadjoint orbits identified with the Entropy level sets. In the framework of Thermodynamics, we associate a symplectic bifoliation structure to describe non-dissipative dynamics on symplectic leaves (on level sets of Entropy as constant Casimir function on each leaf), and transversal dissipative dynamics, given by Poisson transverse structure (Entropy production from leaf to leaf). The symplectic foliation orthogonal to the level sets of moment map is the foliation determined by hamiltonian vector fields generated by functions on dual Lie algebra. The orbits of a Hamiltonian action and the level sets of its moment map are polar to each other. The space of Casimir functions on a neighborhood of a point is isomorphic to the space of Casimirs for the transverse Poisson structure. Souriau's model could be then interpreted by Mademoiselle Paulette Libermann's foliations, clarified as dual to Poisson Gamma-structure of Haefliger, which is the maximum extension of the notion of moment in the sense of J.M. Souriau, as introduced by P. Molino, M. Condevaux and P. Dazord in papers of "Séminaire Sud-Rhodanien de Geometrie . The symplectic duality to a symplectically complete foliation, in the sense of Libermann, associates an orthogonal foliation. Paulette Libermann proved that a Legendre foliation on a contact manifold is complete if and only if the pseudo-orthogonal distribution is completely integrable, and that the contact form is locally equivalent to the Poincaré-Cartan integral invariant. Paulette Libermann proved a classical theorem relating to coisotropic foliations, which notably gives a proof of Darboux's theorem. We conclude with link to Cartan foliation and Edmond Fedida works on Cartan's mobile frame-based foliation. As observed by Georges Reeb "Thermodynamics has long accustomed mathematical physics [see DUHEM P.] to the consideration of completely integrable Pfaff forms: the elementary heat dQ [notation of thermodynamicists] representing the elementary heat yielded in an infinitesimal reversible modification is such a completely integrable form. This point does not seem to have been explored since then."

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