Certification of approximate singularities with a view to the punctual Hilbert scheme

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Computing singular solutions of polynomial systems $f = (f_1, \ldots, f_N) \in \mathbb{C}[x_1 \ldots, x_n]^N$ is a challenging question from a practical numerical point of view.

To address this problem and solve more efficiently non-linear equations with isolated singular solutions, one can analyse the multiplicity structure or inverse system and then exploit the properties of this structure to develop efficient algorithms. This leads to the conceptualisation of families of multiplicity structures of a given length as an algebraic variety, and to the study of the so-called punctual Hilbert schemes.

In this presentation we take this approach. We first recall algorithms to compute the inverse system of an isolated singular point. We localise our study to the algebraic variety Hilb_B of inverse systems, which admits a given dual (monomial) basis B. We present some algebraic and geometric properties of this variety, which relies on the algorithmic construction of its points.

Using deformations on the punctual Hilbert scheme, we describe a method to certify that an approximate numerical solution of f is close to an isolated singular point of a nearby polynomial system with a prescribed multiplicity structure. It involves Newton iterations applied to an extended deflated system that locally converges, under regularity conditions, to a small deformation of f such that this deformed system has an exact singular solution. The iteration simultaneously converges to the coordinates of the singular solution and the coefficients of the inverse system that describes the multiplicity structure at the singular solution. We use α-theory test to certify the quadratic convergence, and to give bounds on the size of the deformation and on the distance to the singular point.

This is based on joint works with Agnes Szanto, to whom this presentation is dedicated.