Potential Theory with Multivariate Kernels on the Sphere

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While a wealth of theory has been developed for classical energies for kernels with two inputs (such as Riesz kernels), which model pairwise interactions between particles, on the sphere, there as been little to no systematic study of multivariate energies, defined by interactions of triples, quadruples, or even higher numbers of particles, i.e. functionals of the form

$$E_K(\omega_N) = \sum_{j_1 \in \omega_N} \sum_{j_2 \in \omega_N \setminus \{j_1\}} \cdots \sum_{j_n \in \omega_N \setminus \{j_1, \ldots, j_{n-1}\}} K(z_{j_1}, ..., z_{j_n})$$

for finite point sets $\omega_N = \{z_1, ..., z_N\} \subset \mathbb{S}^d$ and

$$I_K(\mu) = \int_{\mathbb{S}^d} \cdots \int_{\mathbb{S}^d} K(x_1, ..., x_n) d\mu(x_1) ... d\mu(x_n),$$

for Borel probability measures μ, with $n \geq 3$ and $K : (\mathbb{S}^d)^n \to \mathbb{R} \cup \{\infty\}$. While less common than energies involving a two-input kernel, multivariate energies have appeared in various settings, including geometric measure theory, material science, and discrete geometry. Generally, one intends to determine minimizers of these energies, or various properties of minimizers.

For two-input rotationally-invariant kernels, one can decompose K into a sum of Gegenbauer polynomials and study the coefficients to determine if the uniform measure minimizes I_K. Likewise, one can use properties of Gegenbauer polynomials to find bounds for the energy via linear programming. We shall discuss recent work to develop analogues of these results for kernels with three or more inputs, and some consequences.

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