On Normal Forms in Differential Galois Theory for the Classical Groups

Matthias Seiß
Universität Kassel, Germany
mseiss@mathematik.uni-kassel.de

In classical Galois theory there is the well-known construction of the general polynomial equation over $\mathbb{Q}$ with Galois group the symmetric group S_n. Shortly recalling this construction we consider a rational function field $\mathbb{Q}(\boldsymbol{x})$ in n indeterminates $\boldsymbol{x} = (x_1, \dots, x_n)$ over $\mathbb{Q}$ on which the symmetric group acts by permuting the variables $\boldsymbol{x}$. The fix field under this action is generated over $\mathbb{Q}$ by the elementary symmetric polynomials

$$\boldsymbol{s}(\boldsymbol{x}) = (s_1(\boldsymbol{x}), \dots, s_n(\boldsymbol{x}))$$

and has transcendence degree n over $\mathbb{Q}$. Moreover, $\mathbb{Q}(\boldsymbol{x})$ is a Galois extension of $\mathbb{Q}(\boldsymbol{s}(\boldsymbol{x}))$ with Galois group S_n and its defining equation is the polynomial equation of degree n whose coefficients are (up to sign) the polynomials $\boldsymbol{s}(\boldsymbol{x})$. Every algebraic extension of $\mathbb{Q}$ which is defined by a polynomial of degree n is obtained as a specialization by substituting the roots for $\boldsymbol{x}$ in the general equation.

In differential Galois theory there is a similar construction of a general differential equation with differential Galois group the general linear group GL_n. More precisely, for an algebraically closed field C of characteristic zero one considers here a differential field $C\langle \boldsymbol{y} \rangle$ which is generated by n differential indeterminates $\boldsymbol{y} = (y_1, \dots, y_n)$. Now the group $\mathrm{GL}_n(C)$ acts on $C\langle \boldsymbol{y} \rangle$ by linearly transforming the indeterminates $\boldsymbol{y}$. For a new differential indeterminate Y the general differential equation is defined as the quotient of the two Wronskians

$$\frac{wr(Y, y_1, \dots, y_n)}{wr(y_1, \dots . y_n)} =: Y^{(n)} + c_{n-1}(\boldsymbol{y})Y^{(n-1)} + \cdots + c_0(\boldsymbol{y})Y = 0.$$

The coefficients $\boldsymbol{c} = (c_{n-1}(\boldsymbol{y}), \dots, c_0(\boldsymbol{y}))$ of the general equation are differentially algebraically independent over C and as the elementary symmetric polynomials they generate the fixed field of $C\langle \boldsymbol{y} \rangle$ under $\mathrm{GL}_n(C)$ over the constants. Clearly, $C\langle \boldsymbol{y} \rangle$ is a Picard-Vessiot extension of $C\langle \boldsymbol{c} \rangle$ with differential Galois group $\mathrm{GL}_n(C)$ and it is generic extension. Indeed, let E be a Picard-Vessiot extension of a differential field F with constants C defined by a linear scalar differential equation of order n. If $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)$ are the linearly independent solutions of this equation, then E/F is obtained as a specialization of the general equation by substituting the solutions $\boldsymbol{\eta}$ for the indeterminates $\boldsymbol{y}$. Generalizations to groups other than $\mathrm{GL}_n(C)$ were obtained in [1] and [2]. In all these cases the general differential equation involves n differential indeterminates over C (apart from Y).

An analogue construction of a general linear differential equation for the classical groups was presented in [5]. This approach combines the geometric structure of a classical group $G(C)$ of Lie rank l with Picard-Vessiot theory. As in the case of the general equation with group $\mathrm{GL}_n(C)$ the construction starts with a differential field $C\langle \boldsymbol{v} \rangle$ generated by l differential indeterminates $\boldsymbol{v} = (v_1, \dots, v_l)$ over C, but this time the general extension field is a Liouvillian extension $\mathcal{E}$ of $C\langle \boldsymbol{v} \rangle$ with differential Galois group a fixed Borel group $B^-(C)$ of $G(C)$. Choosing a Chevalley basis of the Lie algebra $\mathfrak{g}(C)$ of $G(C)$ the Liouvillian extension is defined by a matrix $A_{\mathrm{Liou}}(\boldsymbol{v})$ which is the sum of the Cartan subalgebra parametrized by the indeterminates $\boldsymbol{v}$ and the basis elements of the root spaces corresponding to the negative simple roots. In order to define a group action of $G(C)$ on $\mathcal{E}$ one needs to construct a fundamental matrix $\mathcal{Y}$ and let $G(C)$ act on it by right multiplication which will then induce an action of $G(C)$ on $\mathcal{E}$. To this end, let b be a fundamental matrix of $A_{\mathrm{Liou}}(\boldsymbol{v})$ in $B^-(\mathcal{E})$, $\overline{w}$ a representative of the longest Weyl group element and let $u(\boldsymbol{v}, \boldsymbol{f})$ be the product of matrices of all negative root groups where the matrices corresponding to the negative simple roots are parametrized by $\boldsymbol{v}$ and the matrices corresponding to all remaining negative roots depend on differential polynomials $\boldsymbol{f}$ in $C\{\boldsymbol{v}\}$. These differential polynomials are chosen in such a way that the logarithmic derivative of $\mathcal{Y} = u\overline{w}b$ is the matrix $A_G(\boldsymbol{s}(\boldsymbol{v}))$ constructed in [4] and [5], where

$\boldsymbol{s}(\boldsymbol{v})$ are l differential polynomials in $C\{\boldsymbol{v}\}$. Analogue to the cases of the symmetric group and $\mathrm{GL}_n(C)$ presented above, the $\boldsymbol{s}(\boldsymbol{v})$ are differentially algebraically independent over C. Multiplying $\mathcal{Y}$ from the right with elements of $G(C)$ and then recomputing the Bruhat decomposition of the product defines an action on $\boldsymbol{v}$, $\boldsymbol{f}$ and on the generators of the Liouvillian extension, i.e. the entries of b, and so induces an action of $G(C)$ on $\mathcal{E}$. The fixed field under this induced action is $C\langle \boldsymbol{s}(\boldsymbol{v})\rangle$ and one can show that the field $\mathcal{E}$ is a Picard-Vessiot extension of $C\langle \boldsymbol{s}(\boldsymbol{v})\rangle$ with differential Galois group $G(C)$ for the differential equation defined by $A_G(\boldsymbol{s}(\boldsymbol{v}))$. The construction is only generic for Picard-Vessiot extensions of F with defining matrix gauge equivalent to a matrix in *normal form*, i.e. a specialization of $A_G(\boldsymbol{s}(\boldsymbol{v}))$. Deciding such a gauge equivalence is non-trivial as a consequence of the fact that $\mathcal{E}$ and $C\langle \boldsymbol{s}(\boldsymbol{v})\rangle$ have differential transcendence degree l over C.

This talk is dedicated to the question of the genericity properties of the extension $\mathcal{E}$ over $C\langle \boldsymbol{s}(\boldsymbol{v})\rangle$. We consider the problem of gauge equivalence of a generic element of the Lie algebra to a matrix in normal form. More precisely, let d be the dimension of the classical group G and let $\boldsymbol{a} = (a_1, \ldots, a_d)$ be differential indeterminates over a differential field F with constants C. Further let $A(\boldsymbol{a})$ be a generic element in the Lie algebra $\mathfrak{g}(F\langle \boldsymbol{a}\rangle)$ obtained from parametrizing the Chevalley basis from above with the indeterminates $\boldsymbol{a}$. It is known (cf. [3]) that the differential Galois group of $\boldsymbol{y}' = A(\boldsymbol{a})\boldsymbol{y}$ over $F\langle \boldsymbol{a}\rangle$ is $G(C)$. We present the construction of a differential field extension $\mathcal{L}$ of $F\langle \boldsymbol{a}\rangle$ such that the field of constants of $\mathcal{L}$ is C, the differential Galois group of $\boldsymbol{y}' = A(\boldsymbol{a})\boldsymbol{y}$ over $\mathcal{L}$ is still the full group $G(C)$ and $A(\boldsymbol{a})$ is gauge equivalent over $\mathcal{L}$ to a specialization of $A_G(\boldsymbol{s}(\boldsymbol{v}))$, i.e. to a matrix in normal. In the special case of $G = \mathrm{SL}_3$ we show how one obtains an analogous result for specializations of the coefficients of $A(\boldsymbol{a})$.

References

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