Generalized polynomials and generalized Gaussian quadrature

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Orthogonal polynomials are linearly independent functions, which form a so-called Chebyshev set. Chebyshev sets are defined by uniqueness of interpolation: like polynomials, a Chebyshev set with n functions has a unique solution to the interpolation problem in any set of n distinct points. For this reason, expansions in Chebyshev sets are sometimes called generalized polynomials. It turns out that the unique interpolation property is sufficient to define Gaussian quadrature, in this context called generalized Gaussian quadrature, for the numerical evaluation of integrals. The rules are Gaussian in the sense that with just n points they are exact for twice as many basis functions. Surprisingly, one does not need orthogonality or other algebraic properties of polynomials for this result to be true. For classical Gaussian quadrature, the points are the roots of an orthogonal polynomial. What are they in the case of generalized polynomials? We describe a theoretical framework and formulate a practical and convergent algorithm to compute the rules. We illustrate the method with novel numerical examples of generalized Gaussian quadrature and applications.